Rutgers University: Real Variables and Elementary Point-Set Topology Qualifying Exam

August 2014: Problem 1 Solution

Exercise. Let A be a subset of $[0,1]$. Let m^+ be Lebesgue outer measure on $[0,1]$.

(a) State the definition of "Lebesgue measurable set".

Solution.

A is **Lebesgue measurable** if $\forall E \in \mathcal{P}(\mathbb{R})$,

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A^C)$$

(b) Show that A is Lebesgue measurable if and only if

$$m^*(A) + m^*(A^C) = 1,$$

where A^C is the complement of A in $[0,1]$.

Solution.

$(\implies)$ If A is Lebesgue measurable then

$$\begin{aligned} m^*(A) + m^*(A^C) &= m^*(A \cap [0,1]) + m^*(A^C \cap [0,1]) \\ &= m^*([0,1]) && \text{since } A \text{ Leb. measurable} \\ &= 1 \end{aligned}$$

$(\impliedby)$Let m^* be an outer measure on X. If $E \subset X$, define $m_*(E) = m_0(X) - m^*(E)$, where m_0 is the inner measure. Then E is m^*- measurable $\iff$ $m^*(E) = m_*(E)$.

$$\begin{aligned} & m_*(A) = m([0,1]) - m^*(A^C) \\ \implies \quad & m_*(A) + m*(A^C) = 1 \\ \text{and} \quad & m^*(A) + m^*(A^C) = 1 && \text{(Given)} \\ \implies \quad & m^*(A) = m_*(A) \\ \implies \quad & A \text{ is } m^* - \text{measurable.} \end{aligned}$$